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A Method for Revising the Potential Inconsistent Elements in an Intuitionistic Fuzzy Preference Relation

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Abstract: In this paper, we propose a method that can improve a multiplicative inconsistency by revising the potential inconsistent elements of an intuitionistic fuzzy preference relation (IFPR) without constructing a multiplicative consistent IFPR. After converting the given IFPR into a positive reciprocal matrix based on multiplicative consistency, the necessary and sufficient conditions for the IFPR to be multiplicative consistent or inconsistent put forward. A symmetric deviation matrix that can take accurate measurement of consistency bias of every element in an IFPR is constructed. Which of elements in the IFPR corresponding to the largest bias in the deviation matrix are really inconsistent, is verified by a bias verifying vector and a new method of eliminating alternatives, and are uniquely determined by using the fact that all the determinacy degrees of the IFPR remain constant in the revising process. The proposed method can preserve most information of the original IFPR as well as need a few operations in comparison with previous methods because they require to calculate underlying priority weights of alternatives based on a model. Meanwhile an associated example is offered to show the correctness and efficiency of the proposed method.

Keywords: Multiplicative Consistency, Determinacy Degree, Symmetric Deviation Matrix, Bias Verifying Vector, Method of Eliminating Alternatives

1. Introduction

In most real decision making problems, decision makers (DMs) may not provide their preferences over the alternatives with exact nuerical values because of lack of precise or sufficient level of knowledge related to the problem domain, or difficulty in explaining explicitly the degree to which one alternative is better than others. In this situation, there is usually some indeterminacy between the alternatives considered. Although traditional fuzzy sets introduced by Zadeh [32] can be used to represent the fuzzy and indeterminacy preferences of DMs in the process of a decision making, its applications are limited. Since the non-membership degree of every element in Zadeh's fuzzy set is expressed by the complement of its membership, this actually ignores the decision maker's indeterminacy. The advantage of IFPR [17, 26] by Atanassov's intuitionistic fuzzy set [1] is the capability of representing inevitably imprecise or not totally reliable judgments and capability of expressing indeterminacy degree with the help of membership degree and non- membership degree. Due to its flexibility in handling vagueness/indeterminacy, intuitionistic fuzzy sets have been extensively used in many areas, such as fuzzy logics [9], fuzzy cognitive map [16], pattern recognition [21] and decision making [10, 14, 26, 28, 29]. For example, Fujita and Hakura have used intuitionistic fuzzy sets to represent and model medical doctor responses in medical diagnosis as part of the mental cloning ''used to mirror a person cognitive behaviour into a model that interacts with human users'' [4] on building the virtual doctor system (VDS) for medical applications [5, 6].

A very important research topic for an IFPR is how to judge its consistency. The lack of consistency that guarantees a transitivity of preferences may lead to inconsistent or irrational results for decision making problems. Thus, consistencies of IFPRs require that all the preferences bring about no contradiction. At present, there are mainly two types of consistency concepts for IFPRs: additive consistency and multiplicative consistency. Additive consistency is, to some extent, inappropriate in modeling consistency, due to that its condition is sometimes in conflict with the scale used for providing preference values, but multiplicative consistency does not have this limitation [2]. There are several different forms of definition for multiplicative consistent IFPRs [10, 12, 22, 24, 25, 27, 29]. By using these consistency concepts of IFPRs, many mathematical programming models for deriving the underlying priority weight's vectors from a IFPR are developed to improve inconsistency [7, 10, 12, 13, 15, 23, 24, 28, 30]. If an IFPR is consistent, all the priority weight's vectors obtained by different models are the same. However, if an IFPR is inconsistent, the priority weight's vectors are different, and this has a direct impacts on the ranking results of the final decision. Hence, many studies have been focused on inconsistency improving problems [8, 11, 13-15, 19, 28, 30, 31]. In particular, there are automatic procedure [13, 28] and iterative algorithm [14, 30] that improve the multiplicative inconsistency of an IFPR. These methods require to calculate the underlying priority weghts of alternatives based on a given IFPR. In addition, Meng et al. [14] proposed a concept of the multiplicative consistency of an IFPR by using preferred IFPR and constructed 0-1 mixed programming models to improve the multiplicative inconsistency. This method seems to be too more complex than the previous ones. Hyonil et al. [8] proposed a method to improve the multiplicative inconsistency of an IFPR based on indeterminacy degrees of one.

Motivation: Based on investigation of previous methods to improve multiplicative inconsistent IFPRs, we discover that they still have a shortcomming: First, since models to derive the underlying priority weight's vector in connection with alternatives are nonlinear their acceptable solutions may not exist. Accodingly, the multiplicative cosistent IFPR can not calculate. Second,. it is yet pending which of underlying priority weight's vectors derived from an inconsistent IFPR by various methods or models is really correct. Next, the larger the distance deviation between the given IFPR and the multiplicative consistent IFPR obtained from one, the greater is difference of the result derived from an acceptable consistent IFPR from DM's opinion.. For this reason, we propose a method that can improve the multiplicative inconsistency by revising the potential inconsistent elements in an IFPR based on a new method of eliminating alternatives without calculating any underlying priority weight's vector to construc a multiplicative consistent IFPR.

To do this, the rest of the paper is organized as follows: Section 2 review the mathematical frameworks for preference relations and the results associated with their consistencies. In section 3, we suggest and prove necessary and sufficient conditions for an IFPR to be multiplicative consistent or inconsistent. Section 4 constructs a symmetric deviation matrix and proposes an algirithm that can verify the potential inconsistent elements by using a bias verifying vector and a new method of eliminating alternatives, and uniquely determine the inconsistent ones. The paper ends with the conclusion in Section 5.

2. Preliminary

Let $X=\{x_1,x_2,...,x_n\}$ be a set of $n(\geq 3)$ alternatives. DMs compare each pair (x_i,x_j) of alternatives so as to express their opinions or preferences on such set.

Definition 2.1. A preference relation P on such a set X is a binary relation $\mu_P: X\times X\to D$, where D is the domain of representation of preference degrees provided by DMs.

A preference relation P may be conveniently represented by a matrix $P=(p_{ij})_{n\times n}$ of dimension $card(X)$, with $p_{ij}=\mu_P(x_i,x_j)$ being interpreted as the degree or intensity of preference of alternative x_i over x_j. The elements of P could represent numeric or linguistic, respectively. In this contribution, we are going to focus on fuzzy preference relations [3] and the intuitionistic fuzzy preference relations [26].

Definition 2.2. A fuzzy preference relation $B=(b_{ij})_{n\times n}$ on the set X is a binary relation in $X\times X$, that is characterized by a membership function $\mu_B: X\times X\to[0,1]$ with the following interpretation:

$$\mu_B(x_i,x_j)=b_{ij},\ b_{ij}+b_{ji}=1,\ i,j=1,2,...,n, \quad (1)$$

where b_{ij} indicates the degree that the alternative x_i is preferred to alternative x_j.

To model DM's pairwise comparisons with indeterminacy, Xu [26] introduced the concept of IFPRs.

Definition 2.3. An intuitionistic fuzzy preference relation $R=(r(x_i,x_j))_{n\times n}$ $(r(x_i,x_j)=(\mu(x_i,x_j),\nu(x_i,x_j))$ on the set X is characterized by a membership function $\mu: X\times X\to[0,1]$ and a non-membership function $\nu: X\times X\to[0,1]$ such that

$$0\leq\mu(x_i,x_j),\nu(x_i,x_j)\leq 1,\ \mu(x_i,x_i)=\nu(x_i,x_i)=0.5,\ \mu(x_i,x_j)=\nu(x_j,x_i),\ \ \mu(x_i,x_j)+\nu(x_i,x_j)\leq 1,\ i,j=1,2,\cdots,n,$$

where $\mu(x_i,x_j)$ denotes the certainty degree to which the alternative x_i is preferred to the alternative x_j, and $\nu(x_i,x_j)$ indicates the certainty degree to which the alternative x_i is not preferred to the alternative x_j. For simplicity, let's denote $r(x_i,x_j)=r_{ij}$, $\mu(x_i,x_j)=\mu_{ij}$ and $\nu(x_i,x_j)=\nu_{ij}$. Especially, $\pi_{ij}=1-\mu_{ij}-\nu_{ij}$ is the indeterminacy degree that represents an amount of lacking information in determining the membership degree and non-membership degree between alternatives x_i and x_j, and the value of $\mu_{ij}+\nu_{ij}$ is called the determinacy degree.

In comparison with fuzzy preference relations, the use of IFPRs in decision making is limited, which is mainly due to the computational complexity associated to using membership degree, non-membership degree and indeterminacy degree to model experts' subjective preferences. To overcome the complexity problem, Ureña et al. [20] proved the fact that IFPR $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ is mathematically isomorphic with asymmetric fuzzy preference raltion $R=(r_{ij})_{n\times n}(r_{ij}=\mu_{ij})$:

$$0\le\mu_{ij}\le 1,\ 0\le\mu_{ij}+\mu_{ji}\le 1,\ \mu_{ii}=0.5,\quad i,j=1,2,\cdots,n. \quad (2)$$

However, the total amount of computation does not decrease even though the form of preferences is simple.

Consistency of fuzzy preference relations has been modeled using the notion of transitivity in the pairwise comparison among any three alternatives [18].

Definition 2.4. A fuzzy preference relation $B=(b_{ij})_{n\times n}$ is multiplicative consistency if the following multiplicative transitivity is satisfied:

$$b_{ij}b_{jk}b_{ki}=b_{ik}b_{kj}b_{ji},\ i,j=1,2,\cdots,n. \quad (3)$$

Liao and Xu [12] formulated a definition for the multiplicative consistency of IFPRs.

Definition 2.5. An IFPR $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ is multiplicative consistent if the following transitivity is satisfied:

$$\mu_{ij}\mu_{jk}\mu_{ki}=\nu_{ij}\nu_{jk}\nu_{ki},\ i,j,k=1,2,\cdots,n. \quad (4)$$

Then, the multiplicative consistency of an asymmetric fuzzy preference relation $R=(r_{ij})_{n\times n}\ (r_{ij}=\mu_{ij})$ is defined as follows:

$$\mu_{ij}\mu_{jk}\mu_{ks}=\mu_{sk}\mu_{kj}\mu_{ji},\ i,j,k=1,2,\cdots,n. \quad (5)$$

Liao and Xu [12] provided a transformation formula to convert the normalized intuitionistic fuzzy priority weight vector $w=(w_1,w_2,...,w_n)^T$ into a multiplicative consistent IFPR $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ in the sense of Definition 2.5:

$$r_{ij}=(\mu_{ij},\nu_{ij})=\begin{cases}(0.5,0.5) & i=j\\ \left(\dfrac{2w_i^{\mu}}{w_i^{\mu}-w_i^{\nu}+w_j^{\mu}-w_j^{\nu}+2},\dfrac{2w_j^{\mu}}{w_i^{\mu}-w_i^{\nu}+w_j^{\mu}-w_j^{\nu}+2}\right) & i\ne j\end{cases}, \quad (6)$$

where

$$w_i^{\mu},\ w_i^{\mu}\in(0,1),\ w_i^{\mu}+w_i^{\mu}\le 1,\ \sum_{i=1,i\ne j}^{n}w_i^{\mu}\le w_j^{\nu},w_j^{\mu}+n-2\ge\sum_{i=1,i\ne j}^{n}w_i^{\nu},\ \ i,j=1,2,\cdots,n.$$

Based on the transformation formula, they constructed the following fractional programming model to derive the normalized intuitionistic fuzzy priority weight's vector:

$$Min\ Z=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\varepsilon_{ij}^{+}+\varepsilon_{ij}^{-}+\xi_{ij}^{+}+\xi_{ij}^{-})$$

$$s.t.\begin{cases}\dfrac{2w_i^{\mu}}{w_i^{\mu}-w_i^{\nu}+w_j^{\mu}-w_j^{\nu}+2}-\mu_{ij}-\varepsilon_{ij}^{+}+\varepsilon_{ij}^{-}=0,\ i=1,2,\cdots,n-1;\\ \qquad j=1,2,\cdots,n\\ \dfrac{2w_j^{\mu}}{w_i^{\mu}-w_i^{\nu}+w_j^{\mu}-w_j^{\nu}+2}-\mu_{ij}-\xi_{ij}^{+}+\xi_{ij}^{-}=0,\ i=1,2,\cdots,n-1;\\ \qquad j=1,2,\cdots,n\\ w_i^{\mu},\ w_i^{\nu}\in[0,1],\ i=1,2,\cdots,n\\ \sum_{j=1,\,j\ne i}^{n}w_j^{\mu}\le w_i^{\nu},\ w_i^{\mu}+n-2\ge\sum_{j=1,\,j\ne i}^{n}w_j^{\nu},\ i=1,2,\cdots,n\\ \varepsilon_{ij}^{+},\ \varepsilon_{ij}^{-}\ge 0,\ \xi_{ij}^{+},\ \xi_{ij}^{-}\ge 0,\ i=1,2,\cdots,n-1;\ j=1,2,\cdots,n\end{cases}. \quad (7)$$

After calculating the priority weight's vector from a given IFPR by model (7), the multiplicative consistent IFPR is calculated through Eq. (6).

Due to the complexity of a decision making problem and the limited knowledge of DMs, to furnish the multiplicative consistent IFPRs is nearly impossible for DMs, especially when the number of alternatives is too large. Thus, Liao and Xu [14] defined an acceptable multiplicative consistent IFPR.

Definition 2.9. Let $R=(r_{ij})_{n\times n}(r_{ij}=(\mu_{ij},\nu_{ij}))$ be an IFPR; then R is called an acceptable multiplicative consistent if

$$d(R,\overline{R})\le \xi \tag{8}$$

where $d(R,\overline{R})$ is the distance measure between the IFPR R and its corresponding multiplicative consistent $\overline{R}$, which can be calculated by

$$d(R,\overline{R})=\frac{1}{(n-1)(n-2)}\sum_{1\le i<j\le n}^{n}(|\mu_{ij}-\overline{\mu}_{ij}|+|\nu_{ij}-\overline{\nu}_{ij}|+|\pi_{ij}-\overline{\pi}_{ij}|) \tag{9}$$

and ξ is the prescribed consistency threshold.

In order to improve multiplicative inconsistency of IFPR $R=(r_{ij})_{n\times n}$ $(r_{ij}=(\mu_{ij},\nu_{ij}))$, an acceptable consistent IFPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ $(\tilde{r}_{ij}=(\tilde{\mu}_{ij},\tilde{\nu}_{ij}))$ is calculated through the following formula [13, 14]:

$$\tilde{\mu}_{ij}=\mu_{ij}^{1-p\delta}\overline{\mu}_{ij}^{p\delta},\ i,j=1,2,\cdots,n, \tag{10}$$

$$\tilde{\nu}_{ij}=\nu_{ij}^{1-p\delta}\overline{\nu}_{ij}^{p\delta},\ i,j=1,2,\cdots,n, \tag{11}$$

where $\overline{R}=(\overline{r}_{ij})_{n\times n}(\overline{r}_{ij}=(\overline{\mu}_{ij},\overline{\nu}_{ij}))$ is a multiplicative consistent IFPR corresponding to an IFPR R, and p indicates the iteration number and $\delta\in(0,1)$ is the controlling parameter determined by DM. Besides the consistent elements, every elements in the acceptable consistent IFPR $\tilde{R}$ constructed by these methods is very different from it's corresponding elements in the IFPR.

The method proposed by Meng et al. [15] is superficially similar to the formula (10) and (11).

The method by Hyonil et al. [8] makes a great difference from previous method that derived the underlying priority weight's vector with respect to alternatives based on an IFPR. They improved the multiplicative inconsistency by method that revises the most inconsistent elements based on the complete consistent reciprocal matrix $F=(f_{ij})_{n\times n}$ $\left(f_{ij}=\sqrt[n]{\prod_{k=1}^{n}\frac{\mu_{ik}}{\nu_{ik}}\frac{\mu_{kj}}{\nu_{kj}}}\right)$ associated with an IFPR $R=(r_{ij})_{n\times n}$.

3. Conditions Equivalent to Multiplicative Consistent IFPRs

In this section, we propose and prove necessary and sufficient conditions for an IFPR to be multiplicative consistency or inconsistency after converting the IFPR into a positive reciprocal matrix.

From now on, we use intuitionistic fuzzy preference relation by expression $R=(\mu_{ij})_{n\times n}$ instead of $R=(r_{ij})_{n\times n}$ $(r_{ij}=(\mu_{ij},\nu_{ij}))$ based on the fact that the set of IFPRs is mathematically isomorphic with the set of asymmetric fuzzy preference relations.

Definition 3.1. An IFPR $R=(r_{ij})_{n\times n}$ $(r_{ij}=(\mu_{ij},\nu_{ij}))$ is multiplicative consistent if the following condition is satisfied:

$$\frac{\mu_{ij}}{\mu_{ji}}=\frac{\mu_{is}}{\mu_{si}}\frac{\mu_{sj}}{\mu_{js}},\quad i,j,s=1,2,\cdots,n. \tag{12}$$

Let us introduce the following matrix $A=(a_{ij})_{n\times n}$, where $a_{ij}=\frac{\mu_{ij}}{\mu_{ji}}$, $\mu_{ij},\mu_{ji}\in R$. Then the matrix $A=(a_{ij})_{n\times n}$ is reciprocal, i.e., $a_{ij}=\frac{1}{a_{ji}}$ for $i,j=1,2,\cdots,n$.

Definition 3.1 for multiplicative consistency of IFPRs is expressed as follows:

$$a_{ij}=a_{is}a_{sj},\quad i,j,s=1,2,\cdots,n. \tag{13}$$

In addition, we construct a matrix $C=(c_{ij})_{n\times n}$ by using Eq. (13):

$$c_{ij}=\frac{1}{n}\sum_{s=1}^{n}a_{is}a_{sj}\frac{1}{a_{ij}}=\frac{1}{n}\sum_{s=1}^{n}a_{is}a_{sj}a_{ji},\ i,j=1,2,\cdots,n. \tag{14}$$

Based on the matrix C, a necessary and sufficient condition for an IFPR to be multiplicative consistent is proposed.

Theorem 3.1. An IFPR $R=(\mu_{ij})_{n\times n}$ is multiplicative consistent if and only if $C=(1)_{n\times n}$.

Proof. Let R be multiplicative consistent IFPR. From Eq. (13), we have:

$$c_{ij}=\frac{1}{n}\sum_{s=1}^{n}a_{is}a_{sj}\frac{1}{a_{ij}}=\frac{1}{n}\sum_{s=1}^{n}1=1,\ i,j=1,2,\cdots,n.$$

Conversely, we assume that the IFPR $R=(\mu_{ij})_{n\times n}$ is multiplicative inconsistent in spite of that

$$c_{ij} = \frac{1}{n}\sum_{s=1}^{n} a_{is}a_{sj}\frac{1}{a_{ij}} = 1, \quad i,j = 1,2,\cdots,n,$$

thus, $a_{ij} \neq a_{ik}a_{kj}$ for some i,j,k. Adding up all the equations according to $j = 1,2,\cdots,n$ in Eq. (14), we have:

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{s=1}^{n} a_{is}a_{sj}\frac{1}{a_{ij}} = n \Leftrightarrow \sum_{j=1}^{n}\sum_{s=1}^{n} a_{is}a_{sj}\frac{1}{a_{ij}} = n^2.$$

From the above equation, we have:

$$n + \sum_{j>s}\frac{a_{is}a_{sj}}{a_{ij}} + \sum_{j<s}\frac{a_{is}a_{sj}}{a_{ij}} = n^2 \;\Leftrightarrow\; \sum_{j>s}\frac{a_{is}a_{sj}}{a_{ij}} + \sum_{j<s}\frac{a_{is}a_{sj}}{a_{ij}} = n^2 - n$$

$$\Leftrightarrow \sum_{j>s}\left(\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}a_{js}}{a_{is}}\right) = n^2 - n \;\Leftrightarrow\; \sum_{j>s}\left(\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}}{a_{is}a_{sj}}\right) = n^2 - n. \tag{15}$$

On the other hand, since if $x > 0$ then $x + \frac{1}{x} \geq 2$, we have that $\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}}{a_{is}a_{js}} \geq 2$, $i,j = 1,2,\cdots,n$. There are $n(n-1)/2$ terms at the left-hand side of the last equality of Eq. (15). Therefore,

$$\sum_{j>s}\left(\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}}{a_{is}a_{js}}\right) \geq \frac{n(n-1)}{2}\times 2 = n^2 - n. \tag{16}$$

Eq. (16) holds with equality if and only if $\frac{a_{is}a_{sj}}{a_{ij}} = 1$ for all $i,j,s = 1,2,\cdots,n$, i.e., $a_{ij} = a_{is}a_{sj}$, for all $i,j,s = 1,2,\cdots,n$. This contradicts with our assumption that $a_{ij} \neq a_{ik}a_{kj}$, for some i,j,k. As a result, IFPR $R = (\mu_{ij})_{n\times n}$ is multiplicative consistent.

Based on Theorem 3.1, we propose a necessary and sufficient condition for an IFPR to be multiplicative inconsistent.

Theorem 3.2. An IFPR $R = (\mu_{ij})_{n\times n}$ is multiplicative inconsistent if and only if $c_{ij} > 1$ or $c_{ji} > 1$ for some $i \neq j$ respectively, then $0 < c_{ji} < 1$ or $0 < c_{ij} < 1$.

Proof. Since matrix $A = (a_{ij})_{n\times n}$ is reciprocal, we have:

$$c_{ij} + c_{ji} = \frac{1}{n}\sum_{s=1}^{n}\left(\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{js}a_{si}}{a_{ji}}\right) = \frac{1}{n}\sum_{s=1}^{n}\left(\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}}{a_{is}a_{sj}}\right).$$

Since IFPR $R = (\mu_{ij})_{n\times n}$ is multiplicative inconsistent, then we have $\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}}{a_{is}a_{sj}} > 2$, for some $i \neq j \neq s$ due to $x + \frac{1}{x} > 2$ for $x > 0,\ x \neq 1$. Therefore,

$$c_{ij} + c_{ji} = \frac{1}{n}\sum_{s=1}^{n}\left(\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}}{a_{is}a_{sj}}\right) > \frac{1}{n}\times n\times 2 = 2. \tag{17}$$

Eq. (17) is equivalent to $c_{ij} - 1 > 1 - c_{ji}$ or $c_{ji} - 1 > 1 - c_{ij}$ respectively. This means that if $c_{ij} > 1$ or $c_{ji} > 1$ respectively, then $0 < c_{ji} < 1$ or $0 < c_{ij} < 1$.

Necessity is obvious from Theorem 3.1, as a result, we have proved Theorem 3.2.

4. A method for Revising the Potential Inconsistent Elements in an IFPR

In this section, we construct a symmetric deviation matrix and based on them, propose an algirithm that can verify the potential inconsistent elements in an IFPR using a bias verifying vector and a new method of eliminating alternatives, and uniquely determine the inconsisten ones from the fact that all the determinacy degrees in the IFPR are constant in the revisiing process. The correctness and efficiency of proposed method are illustrated with an example.

The construction of a multiplicative consistent IFPR is rather exceptional than usual, due to the complexity of a problem, time pressure, or lack of knowledge about problem domain.

If we introduce the following expression

$$e_{ij}^{(s)} = \max\{\frac{a_{is}a_{sj}}{a_{ij}}, \frac{a_{ij}}{a_{is}a_{sj}}\}, \tag{18}$$

then $\frac{1}{e_{ij}^{(s)}} = \min\{\frac{a_{is}a_{sj}}{a_{ij}}, \frac{a_{ij}}{a_{is}a_{sj}}\}$. Here $e_{ij}^{(s)} \geq 1$, $\frac{1}{e_{ij}^{(s)}} \leq 1$, and we consider the following matrix $D = (d_{ij})_{n\times n}$:

$$d_{ij}^{(s)} = \left(e_{ij}^{(s)} - 1\right) - \left(1 - \frac{1}{e_{ij}^{(s)}}\right) = \frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}}{a_{is}a_{sj}} - 2, \; d_{ij} = \frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,j}}^{n} d_{ij}^{(s)}. \tag{19}$$

If an IFPR $R = (\mu_{ij})_{n\times n}$ is multiplicative consistent, we have $d_{ij} = d_{ij}^{(s)} = 0$ for $i, j, s = 1, 2, \cdots, n$. Otherwise, there exist elements in the matrix D larger than zero, due to the construction of D.

In connection with $d_{ij}^{(s)}$, $s = 1, 2, \cdots, n$, the following theorem is formulated.

Theorem 4.1. The value of $d_{ij}^{(k)}$ is the greatest of $d_{ij}^{(s)}$, $s = 1, 2, \cdots, n,\ s \neq k$ if and only if the value of $e_{ij}^{(k)} - 1$ is the greatest of $e_{ij}^{(s)} - 1$, $s = 1, 2, \cdots, n,\ s \neq k$.

Proof. Since $d_{ij}^{(k)}$ is the greatest of $d_{ij}^{(s)}$, $s = 1, 2, \cdots, n$, we have:

$$d_{ij}^{(k)} - d_{ij}^{(s)} = \left(e_{ij}^{(k)} + \frac{1}{e_{ij}^{(k)}} - 2\right) - \left(e_{ij}^{(s)} + \frac{1}{e_{ij}^{(s)}} - 2\right) = \left(e_{ij}^{(k)} - e_{ij}^{(s)}\right) - \frac{e_{ij}^{(k)} - e_{ij}^{(s)}}{e_{ij}^{(k)}e_{ij}^{(s)}} \geq 0,$$

$$s = 1, 2, \cdots, n,\ s \neq k \quad \Leftrightarrow \quad e_{ij}^{(k)} - e_{ij}^{(s)} \geq \frac{e_{ij}^{(k)} - e_{ij}^{(s)}}{e_{ij}^{(k)}e_{ij}^{(s)}}, \; s = 1, 2, \cdots, n,\ s \neq k.$$

If the value of $e_{ij}^{(k)}$ is smaller than the value of $e_{ij}^{(s)}$ for some s, there is $e_{ij}^{(k)} - e_{ij}^{(s)} < \frac{e_{ij}^{(k)} - e_{ij}^{(s)}}{e_{ij}^{(k)}e_{ij}^{(s)}}$ due to $e_{ij}^{(k)},\ e_{ij}^{(s)} \geq 1$. This inequality contradicts the expressions $e_{ij}^{(k)} - e_{ij}^{(s)} \geq \frac{e_{ij}^{(k)} - e_{ij}^{(s)}}{e_{ij}^{(k)}e_{ij}^{(s)}}$, $s = 1, 2, \cdots, n$. So, the value of $e_{ij}^{(k)} - 1$ is the greatest of $e_{ij}^{(s)} - 1$, $s = 1, 2, \cdots, n,\ s \neq k$.

Conversely, we assume that the value of $e_{ij}^{(k)} - 1$ is the greatest of $e_{ij}^{(s)} - 1$, $s = 1, 2, \cdots, n,\ s \neq k$. Since $e_{ij}^{(k)},\ e_{ij}^{(s)} \geq 1$, $s = 1, 2, \cdots, n,\ s \neq k$, we have:

$$e_{ij}^{(k)} - e_{ij}^{(s)} \geq \frac{e_{ij}^{(k)} - e_{ij}^{(s)}}{e_{ij}^{(k)}e_{ij}^{(s)}}, \; s = 1, 2, \cdots, n \quad \Leftrightarrow \quad \left(e_{ij}^{(k)} - e_{ij}^{(s)}\right) - \frac{e_{ij}^{(k)} - e_{ij}^{(s)}}{e_{ij}^{(k)}e_{ij}^{(s)}} \geq 0$$

$$\Leftrightarrow \quad \left(e_{ij}^{(k)} + \frac{1}{e_{ij}^{(k)}} - 2\right) - \left(e_{ij}^{(s)} + \frac{1}{e_{ij}^{(s)}} - 2\right) \geq 0 \quad \Leftrightarrow \quad d_{ij}^{(k)} \geq d_{ij}^{(s)}, \; s = 1, 2, \cdots, n.$$

As a result, Theorem 4.1 has been proved.

To reduce the computational amount of D, we introduce the following theorem.

Theorem 4.2. The matrix $D = (d_{ij})_{n\times n}$ is non-negative symmetric.

Proof. Since the matrix $A = (a_{ij})_{n\times n}$ is symmetric, we have:

$$d_{ij} = \frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,j}}^{n}\left(\frac{a_{is}a_{sj}}{a_{ij}} + \frac{a_{ij}}{a_{is}a_{sj}} - 2\right) = \frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,j}}^{n}\left(\frac{\frac{1}{a_{js}a_{si}}}{\frac{1}{a_{ji}}} + \frac{\frac{1}{a_{ji}}}{\frac{1}{a_{js}a_{si}}} - 2\right) = \frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,j}}^{n}\left(\frac{a_{js}a_{si}}{a_{ji}} + \frac{a_{ji}}{a_{js}a_{si}} - 2\right) = d_{ji}.$$

In addition, $a_{ij}=\dfrac{\mu_{ij}}{\mu_{ji}}>0$ and $\dfrac{a_{is}a_{sj}}{a_{ij}}+\dfrac{a_{ij}}{a_{is}a_{js}}\geq 2$ for $i,j,s=1,2,\cdots,n$, so $d_{ij}\geq 0$. As a result, Theorem 4.2 has been proved.

Definition 4.1. The matrix $D=(d_{ij})_{n\times n}$ is called deviation matrix.

Based on deviation matrix D, the inconsistency index $CI(D)$ of an IFPR $R=(\mu_{ij})_{n\times n}$ is defined as follows.

Definition 4.2.

$$CI(D)=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d_{ij}\ . \tag{20}$$

In general, improving the multiplicative inconsistency of an IFPR is due to derive the reasonable result in a decision making not that a new information in connection with alternatives is addded.

Based on the above results, we constitute an algorithm for improving the multiplicative inconsistency of an IFPR.

Algorithm 4,1

1) identify location of the largest bias in the deviation matrix D.

Step 1. After transforming an IFPR $R=(\mu_{ij})_{n\times n}$ into the matrix $A=(a_{ij})_{n\times n}$, calculate the deviation matrix D. If the inconsistency index $CI(D)$ is less than or equal to a prescribed inconsistency threshold τ, then go to Step 10.

Step 2. Identify the locations of the largest bias in D. For instance, suppose d_{ij} is such an element in D, where the location is i^{th} row and j^{th} column.

2) Verify the potential inconsistent elements in a bias verifying vector f.

Step 3. Calculate bias verifying vector f from the i^{th} row vector $a_i=(a_{i1},a_{i2},\cdots,a_{in})$ and j^{th} column vector $a_j^T=(a_{1j},a_{2j},\cdots,a_{nj})^T$ of the matrix A, where T represents the transpose symbol:

$$f=\frac{1}{n-2}\left(f_1,f_2,\ldots,f_n\right)=\frac{1}{n-2}\left(\left(e_{ij}^{(1)}-1\right)-\left(1-\frac{1}{e_{ij}^{(1)}}\right),\ \left(e_{ij}^{(2)}-1\right)-\left(1-\frac{1}{e_{ij}^{(2)}}\right),\right.$$
$$\left.\cdots,\ \left(e_{ij}^{(n)}-1\right)-\left(1-\frac{1}{e_{ij}^{(n)}}\right)\right). \tag{21}$$

Step 4. If more values in the vector f are around zero and fewer values are far away from zero, then select the largest bias f_k in f:

$$f_k=\max\{f_1,f_2,\ \cdots,f_n\}.$$

The larger the value of $f_k=\dfrac{1}{n-2}\left(\left(e_{ij}^{(k)}-1\right)-\left(1-\dfrac{1}{e_{ij}^{(k)}}\right)\right)$ is, the larger is the value of $e_{ij}^{(k)}$, owing to Theorem 4.1. Then one or both of a_{ik}, a_{kj} may be too large.

substep 4.1. Choose a element corresponding to the larger value of d_{ik}, d_{kj}. For instance, let it be a_{ik}. Then the value of a_{ik} is revised as $\dfrac{a_{ij}}{a_{kj}}$. Next, go to Step 7.

substep 4.2. If the values of d_{ik}, d_{kj} are similar to each other, use the method of eliminating alternatives. Next, go to Step 7.

Step 5. If more bias values in f are far away from zero and fewer bias values are around zero, then the value of a_{ij} may be revised as:

$$a_{ij}=\frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,j}}^{n}a_{is}a_{sj}\ . \tag{22}$$

Step 6. If the number of elements that the bias values in f are far away from zero is equal to the number of elements that the bias values in f are around zero, then compare the value of d_{ij} with the largest bias f_k.

substep 6.1. If $d_{ij}\approx f_k$, then one or both of a_{ik} and a_{kj} are too large, due to the expression:

$$d_{ij}-f_k=\frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,j}}^{n}\left(\frac{a_{is}a_{sj}}{a_{ij}}+\frac{a_{ij}}{a_{is}a_{sj}}-2\right)-\frac{1}{n-2}\left(\frac{a_{ik}a_{kj}}{a_{ij}}+\frac{a_{ij}}{a_{ik}a_{kj}}-2\right)$$
$$=\frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,j,k}}^{n}\left(\frac{a_{is}a_{sj}}{a_{ij}}+\frac{a_{ij}}{a_{is}a_{sj}}-2\right)\approx 0.$$

Use the method of eliminating alternatives. Next, go to Step 7.

substep 6.2. If the values of d_{ij} and f_k are different to each other, then the value of a_{ij} should be revised because the value of $d_{ij}-f_k=\frac{2}{n-2}\sum_{\substack{s=1,\\ s\neq i,j,k}}^{n}\left(e_{ij}^s+\frac{1}{e_{ij}^s}-2\right)$ is far away from zero. Then the value of a_{ij} is revised as Eq. (22). Next, go to Step 7.

3) An element being revised in A is uniquely determined by the elements in IFPR from the fact that all the determinacy degrees in IFPR R remain constant in the revising process.

Step 7. Assume that an element a_{ct} in A ought to be revised as $\bar{a}_{ct}$, then determinacy degree of a_{ct} is $\mu_{ct}+\mu_{tc}=\mu\leq 1$. On that occasion, the membership degree $\bar{\mu}_{ct}$ and non-membership degree $\bar{\mu}_{tc}$ of $\bar{a}_{ct}$ are uniquely determined by the following simultaneous equation:

$$\begin{cases}\dfrac{\bar{\mu}_{ct}}{\bar{\mu}_{tc}}=\bar{a}_{ct}\\ \bar{\mu}_{ct}+\bar{\mu}_{tc}=\mu\end{cases}\tag{23}$$

Step 8. The elements μ_{ij} and μ_{ji} in IFPR R are replaced with $\bar{\mu}_{ij}$ and $\bar{\mu}_{ji}$.

Step 9. Go back to Step 1.

Step 10. Output IFPR R.

Corollary 4.1. The proposed method can preserve a lot of information of the original IFPR and save many operations, since previous methods [13, 14, 15, 28, 30] improve the multiplicative inconsistency of the IFPR R through the process that multiplicative consistent IFPR $\bar{R}$ corresponding to the IFPR R is constructed by solving a model based on the IFPR R and an acceptable consistent IFPR $\tilde{R}$ is finally obtained by the combinations of all the corresponding elements of R and $\bar{R}$.

Method of eliminating alternatives

In general, when comparing pairs of alternatives, situations where DMs are able to accurately express their preferences over all the alternatives are the exception rather than the rule, due to lack of precise or sufficient level of knowledge of the whole problem to tackle, or limit of decision maker's capacity, as a consequence, alternatives x_i, x_j or x_k have impacts on some aspects. Hence, we propose a method of eliminating alternatives that can verify the potential inconsistent elements by eliminating alternatives one by one from X. Suppose that a value of ς is prescribed, and the values of $a_{ik}=\frac{\mu_{ik}}{\mu_{ki}}$ and $a_{kj}=\frac{\mu_{kj}}{\mu_{jk}}$ might be either too large or the value of $a_{ij}=\frac{\mu_{ij}}{\mu_{ji}}$ might be too small:

Step 1. Construct the IFPR $R_{n-1}^{(k)}$ after eliminating alternative x_k from X and calculate the matrix $A_{n-1}^{(k)}$, namely, eliminate k^{th} row and k^{th} column from A. Calculate the deviation matrix $D_{n-1}^{(k)}$ from $A_{n-1}^{(k)}$

Substep 1.1. If the value of $CI(D_{n-1}^{(k)})$ is less than or equal to the value of ς, then a_{ij} is consistent, and therfore μ_{ij} and μ_{ji} are right. Next go to Step 2 to investigate a_{ik} and a_{kj}.

Substep 1.2. If the value of $CI(D_{n-1}^{(k)})$ is greater than the value of ς, then a_{ij} is too small, because of $Ed_{ij}^{(k)}=\frac{1}{n-1}\sum_{\substack{s=1,\\ s\neq k}}^{n}\left(\frac{a_{is}a_{sj}}{a_{ij}}+\frac{a_{ij}}{a_{is}a_{sj}}-2\right)>0$. On that occasion, a_{ij} is revised as $\frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,j}}^{n}a_{is}a_{sj}$. Next go to Step 2.

Step 2. Calculate the matrix $A_{n-1}^{(i)}$ and the deviation matrix $D_{n-1}^{(i)}$ after eliminating alternative x_i from X.

Substep 2.1. If the value of $CI(D_{n-1}^{(i)})$ is less than or equal to the value of ς, then a_{kj} is consistent, and therfore μ_{kj} and μ_{jk} are right. Then a_{ik} is revised as $\frac{a_{ij}}{a_{kj}}$ because a_{ij} is consistent or is revised from Step 1.

Substep 2.2. If the value of $CI(D_{n-1}^{(i)})$ is greater than the value of ς, then a_{kj} is inconsistent from the supposition of Method. Next go to Step 3.

Step 3. Calculate the matrix $A_{n-1}^{(j)}$ and the deviation matrix $D_{n-1}^{(j)}$ after eliminating alternative x_j from X.

Substep 3.1. If the value of $CI(D_{n-1}^{(j)})$ is less than or equal to the value of ς, then a_{ik} is consistent, and therfore μ_{ik} and μ_{ki} are right. Then a_{kj} is revised as $\frac{a_{ij}}{a_{ik}}$.

Substep 3.2. If both of $CI(D_{n-1}^{(i)})$, $CI(D_{n-1}^{(j)})$ are greater than the value of ς, then both of a_{ik}, a_{kj} are inconsistent. Then the values of a_{ik}, a_{kj} are revised as $\frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq i,k}}^{n}a_{is}a_{sk}$, $\frac{1}{n-2}\sum_{\substack{s=1,\\ s\neq k,j}}^{n}a_{ks}a_{sj}$ respectively.

Corollary 4.1. The purpose of Method of eliminating alternatives is to reduce the amount of caculations that can improve the multiplicative inconsistency of an IFPR, as possible.

Example 4.1. Assume that a DM provides his/her preference information over a collection of alternatives as the following IFPR:

$$R=\begin{pmatrix}(0.50,0.50) & (0.50,0.37) & (0.23,0.57) & (0.60,0.19)\\(0.37,0.50) & (0.50,0.50) & (0.45,0.25) & (0.40,0.29)\\(0.57,0.23) & (0.25,0.45) & (0.50,0.50) & (0.30,0.28)\\(0.19,0.60) & (0.29,0.40) & (0.28,0.30) & (0.50,0.50)\end{pmatrix}.$$

The above IFPR R can be simplified as follows:

$$R_1=\begin{pmatrix}0.50 & 0.50 & 0.23 & 0.60\\0.37 & 0.50 & 0.45 & 0.40\\0.57 & 0.25 & 0.50 & 0.30\\0.19 & 0.29 & 0.28 & 0.50\end{pmatrix}.$$

Step 1. Transfer R into the matrix A:

$$A_1=\begin{pmatrix}1 & 50/37 & 23/57 & 60/19\\37/50 & 1 & 9/5 & 40/29\\57/23 & 5/9 & 1 & 15/14\\19/60 & 29/40 & 14/15 & 1\end{pmatrix}.$$

Calculate the deviation matrix D:

$$D=\begin{pmatrix}0.000 & 2.239 & 4.818 & 2.863\\ & 0.000 & 2.154 & 1.863\\ & & 0.000 & 2.777\\ & & & 0.000\end{pmatrix},\ CI(D)=2.786>\tau=0.1.$$

Step 2. The largest bias in D is $d_{13}=4.818$ and its locations are 1^{th} row and 3^{th} column.

Step 3. 1^{th} row and 3^{th} column in A are as follows:

$$a_1=(1\ \ 50/37\ \ 23/57\ \ 60/19),\ a_3^T=(23/57\ \ 9/5\ \ 1\ \ 14/15)^T.$$

Calculate the bias verifying vector f:

$$f=(f_1,f_2,f_3,f_4)=(0,\ 4.194,\ \ 0,\ 5.441).$$

The largest bias is $f_4=5.441$.

Step 6. Since the number of elements that the bias values in f are far away from zero is equal to the number of elements that the bias values in f are around zero, then compare the value of d_{ij} with the largest bias f_k.

By Step 6.2, the value of a_{13} is revised as:

$$a_{13}=\frac{1}{2}\sum_{\substack{s=1,\\ s\neq 1,3}}^{4}a_{1s}a_{s3}=2.689.$$

Step 7. By equations

$$\begin{cases}\dfrac{\bar{\mu}_{13}}{\bar{\mu}_{31}}=2.689\\ \bar{\mu}_{13}+\bar{\mu}_{31}=0.8\end{cases}$$

$$\bar{\mu}_{13}=0.583,\ \bar{\mu}_{31}=0.217.$$

Step 8. IFPR R_2 is represented as follows:

$$R_2=\begin{pmatrix}0.500 & 0.500 & 0.583 & 0.600\\0.370 & 0.500 & 0.450 & 0.400\\0.217 & 0.250 & 0.500 & 0.300\\0.190 & 0.290 & 0.280 & 0.500\end{pmatrix}.$$

Step 1. Then matrices A_2 and D_2 are as follows:

$$A_2=\begin{pmatrix}1 & 500/370 & 583/217 & 600/190\\370/500 & 1 & 450/250 & 400/290\\217/583 & 250/450 & 1 & 150/140\\190/600 & 290/400 & 140/150 & 1\end{pmatrix},\ D_2=\begin{pmatrix}0.000 & 0.148 & 0.009 & 0.146\\ & 0.000 & 0.009 & 0.199\\ & & 0.000 & 0.060\\ & & & 0.000\end{pmatrix}.$$

Then $CI(D_2)=0.095<\tau=0.1$.

Step 10. Output IFPR R_2.

Let's check the method proposed in this paper through Eq. (10). Applying Eq. (7) to R_1 represents as follows:

Model 4.1

$$\text{Min}\ \begin{aligned}J=&\varepsilon_{12}^{+}+\varepsilon_{12}^{-}+\varepsilon_{13}^{+}+\varepsilon_{13}^{-}+\varepsilon_{14}^{+}+\varepsilon_{14}^{-}+\varepsilon_{21}^{+}+\varepsilon_{21}^{-}+\varepsilon_{23}^{+}+\varepsilon_{23}^{-}+\varepsilon_{24}^{+}\\&+\varepsilon_{24}^{-}+\varepsilon_{31}^{+}+\varepsilon_{31}^{-}+\varepsilon_{32}^{+}+\varepsilon_{32}^{-}+\varepsilon_{34}^{+}+\varepsilon_{34}^{-}+\varepsilon_{41}^{+}+\varepsilon_{41}^{-}+\varepsilon_{42}^{+}+\varepsilon_{42}^{-}+\varepsilon_{43}^{+}+\varepsilon_{43}^{-}\end{aligned}$$

$$
s.t\begin{cases}
0.50-\dfrac{w_1^{\mu}}{(1-w_1^{\nu})+(1-w_2^{\nu})}-\varepsilon_{12}^{+}+\varepsilon_{12}^{-}=0\\
0.23-\dfrac{w_1^{\mu}}{(1-w_1^{\nu})+(1-w_3^{\nu})}-\varepsilon_{13}^{+}+\varepsilon_{13}^{-}=0\\
0.60-\dfrac{w_1^{\mu}}{(1-w_1^{\nu})+(1-w_4^{\nu})}-\varepsilon_{14}^{+}+\varepsilon_{14}^{-}=0\\
0.37-\dfrac{w_2^{\mu}}{(1-w_2^{\nu})+(1-w_1^{\nu})}-\varepsilon_{21}^{+}+\varepsilon_{21}^{-}=0\\
0.45-\dfrac{w_2^{\mu}}{(1-w_2^{\nu})+(1-w_3^{\nu})}-\varepsilon_{23}^{+}+\varepsilon_{23}^{-}=0\\
0.40-\dfrac{w_2^{\mu}}{(1-w_2^{\nu})+(1-w_4^{\nu})}-\varepsilon_{24}^{+}+\varepsilon_{24}^{-}=0\\
0.57-\dfrac{w_3^{\mu}}{(1-w_3^{\nu})+(1-w_1^{\nu})}-\varepsilon_{31}^{+}+\varepsilon_{31}^{-}=0\\
0.25-\dfrac{w_3^{\mu}}{(1-w_3^{\nu})+(1-w_2^{\nu})}-\varepsilon_{32}^{+}+\varepsilon_{32}^{-}=0\\
0.30-\dfrac{w_3^{\mu}}{(1-w_3^{\nu})+(1-w_4^{\nu})}-\varepsilon_{34}^{+}+\varepsilon_{34}^{-}=0\\
0.19-\dfrac{w_4^{\mu}}{(1-w_4^{\nu})+(1-w_1^{\nu})}-\varepsilon_{41}^{+}+\varepsilon_{41}^{-}=0\\
0.29-\dfrac{w_4^{\mu}}{(1-w_4^{\nu})+(1-w_2^{\nu})}-\varepsilon_{42}^{+}+\varepsilon_{42}^{-}=0,\\
0.28-\dfrac{w_4^{\mu}}{(1-w_4^{\nu})+(1-w_3^{\nu})}-\varepsilon_{43}^{+}+\varepsilon_{43}^{-}=0\\
\dfrac{w_1^{\mu}+w_2^{\mu}}{(1-w_1^{\nu})+(1-w_2^{\nu})}=0.87,\dfrac{w_1^{\mu}+w_3^{\mu}}{(1-w_1^{\nu})+(1-w_3^{\nu})}=0.80,\dfrac{w_1^{\mu}+w_4^{\mu}}{(1-w_1^{\nu})+(1-w_4^{\nu})}=0.79\\
\dfrac{w_2^{\mu}+w_3^{\mu}}{(1-w_2^{\nu})+(1-w_3^{\nu})}=0.70,\dfrac{w_2^{\mu}+w_4^{\mu}}{(1-w_2^{\nu})+(1-w_4^{\nu})}=0.69,\dfrac{w_3^{\mu}+w_4^{\mu}}{(1-w_3^{\nu})+(1-w_4^{\nu})}=0.58\\
0\le w_1^{\mu},w_1^{\nu}\le 1,0\le w_2^{\mu},w_2^{\nu}\le 1,0\le w_3^{\mu},w_3^{\nu}\le 1,0\le w_4^{\mu},w_4^{\nu}\le 1\\
w_1^{\mu}+w_1^{\nu}\le 1,w_2^{\mu}+w_2^{\nu}\le 1,w_3^{\mu}+w_3^{\nu}\le 1,w_4^{\mu}+w_4^{\nu}\le 1
\end{cases}
$$

Solving this model by an appropriate optimization computer package, it follows that optimal objective value is $J^*(R_1)=0.834$ and intuitionistic fuzzy priority weight vector is represented as:

$$\overline{w}=(\overline{w}_1,\overline{w}_2,\overline{w}_3,\overline{w}_4)^T=((0.32,0.68),(0.24,0.68),(0.16,0.70),(0.12,0.748)).$$

Then, multiplicative consistent IFPR $\overline{R}$ is represented by Eq. (6) as follows:

$$\overline{R}=\begin{pmatrix}0.500 & 0.500 & 0.516 & 0.593\\ 0.370 & 0.500 & 0.450 & 0.400\\ 0.258 & 0.258 & 0.500 & 0.308\\ 0.222 & 0.222 & 0.231 & 0.500\end{pmatrix}.$$

In comparison with the multiplicative consistent IFPR $\overline{R}$, the most inconsistent element in R is 1^{th} row and 3^{th} column. This shows that the location of the most inconsistent element in R_1 is equal with the location indicated by the method proposed in this paper. In addition, applying Eq. (7) to R_2 revised by proposed method is $J^*(R_2)=0.142$. Thus, the optimal values are decreased from $J^*(R)=0.834$ to

$J^*(R_2) = 0.142$.

Next, the method for improving the multiplicative consistency developed by Liao et al. [13, 14, 28] will be applied in the same IFPR R and the obtained results will be compared with our proposed method.

Using Eq. (10) to R_1 and $\bar{R}$ at $\delta = 0.9$, the IFPR is obtained as:

$$R(0.9) = \begin{pmatrix} 0.500 & 0.500 & 0.480 & 0.590 \\ 0.370 & 0.500 & 0.450 & 0.400 \\ 0.280 & 0.260 & 0.500 & 0.310 \\ 0.220 & 0.230 & 0.240 & 0.500 \end{pmatrix}.$$

Then, applying Eq. (7) to $R(0.9)$ is $J^*(0.9) = 0.191$.

After transfering $R(0.9)$ into $A(0.9)$, calculate the deviation matrix $D(0.9)$:

$$A(0.9) = \begin{pmatrix} 1 & 50/37 & 12/7 & 59/22 \\ 37/50 & 1 & 45/26 & 40/23 \\ 7/12 & 26/45 & 1 & 31/24 \\ 22/59 & 23/40 & 24/31 & 1 \end{pmatrix},$$

$$D(0.9) = \begin{pmatrix} 0.000 & -0.063 & 0.288 & -0.149 \\ -0.120 & 0.000 & -0.034 & 0.213 \\ -0.221 & 0.325 & 0.000 & -0.005 \\ 0.176 & -0.173 & 0.056 & 0.000 \end{pmatrix}.$$

Then $CI(D(0.9)) = 0.044 < \tau = 0.1$.

Table 1. *Results by differential methods.*

Method	Position of The lagest bias	Inconsistency Index (CI)	Optimal value of an acceptable IFPR (J^*)	Changed elements in R
[13, 14, 28]	(1.3)	$CI(D(0.9)) = 0.044 < \tau$	$J^*(0.9) = 0.191$	μ_{13}, μ_{14}, μ_{31}, μ_{32}, μ_{34}, μ_{41}, μ_{42}, μ_{43}
Paper	(1.3)	$CI(D_2) = 0.095 < \tau$	$J^*(R_2) = 0.142$	μ_{13}, μ_{31}

The larger the value of δ is, the larger are the deviations between the elements of the original R_1 and their corresponding elements of the $R(\delta)$. The result using the method proposed by Liao et al. [13, 14, 28] at $\delta = 0.9$ is similar to the result by our method, but most elements in $R(0.9)$ are very different from their corresponding elements in the original IFPR R_1. However, using our method changes only two elements r_{13}, r_{31} and needs a few operations, since it does not construct a multiplicative consistent IFPR $\bar{R}$ using Eq. (7) from IFPR R_1 and in addition, not compute the combinations of the elements of R with the ones of $\bar{R}$. As a result, our method is a correct and effective in comparison with the previous methods.

5. Conclusion

The construction of the multiplicative consistent IFPR is rather exceptional than usual, due to the complexity of a problem, time pressure, or lack of knowledge about problem domain. The inconsistency improving process is the fundamental and necessary to guarantee the correctness of information. We propose a method that can improve the multiplicative inconsistency by revising the potential inconsistent elements of an IFPR. For this, we present and prove necessary and sufficient conditions for the IFPR to be multiplicative consistent or inconsistent. Next, a symmetric deviation matrix that can accurately measure the deviation of every element in the IFPR not based on the distance between the every element in the IFPR and every element in its corresponding multiplicative consistent IFPR is composed. Which of the elements in R corresponding to the largest bias in the deviation matrix is really inconsistent, is identified by the bias verifying vector and the method of eliminating alternatives, and as a result, is uniquely determined by elements in IFPR from the fact that the determinacy degrees in IFPR remain constant in the revising process. The proposed method can preserve a lot of information of the original IFPR and save many operations in comparison with previous methods because of not being solved any model for deriving any underlying priority weight's vector in connection with alternatives.

However, there are still lots of work to study in the future, including how to aggregate individual IFPRs for reaching group consensus considering DM's ability and consistency of IFPR. In addition, we are going to introduce our method in the fields of decision making, supply chain management, pattern recognition and medical diagnosis, etc.

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